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# Problems and Solutions Concerning Kantorovich Operators

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Most of the conjectures and open problems related to the global approximation by Kantorovich operators are solved.

## 1. INTRODUCTION

In 1973 Berens and Lorentz [4] proved for the Bernstein polynomials

$$B_nf(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k} \qquad (0 \leqslant x \leqslant 1)$$

that $\|\varphi^{-\alpha}(B_nf-f)\|_{C[0,1]} \leqslant Kn^{-\alpha}$ and $\|\Delta_h^2(f)\|_{C(h,1-h)} \leqslant Kh^{2\alpha}$ $(0<\alpha<1)$[1] are equivalent, where $\varphi(x)=x(1-x)$ and

$$\Delta_h^2(f;x) = f(x-h) - 2f(x) + f(x+h).$$

A dual result is due to Lorentz and Schumaker [6] and Ditzian [5], namely, that $\|B_nf-f\|_{C(0,1)} \leqslant Kn^{-\alpha}$ and $\|\varphi^{\alpha}\Delta_h^2(f)\|_{C(h,1-h)} \leqslant Kh^{2\alpha}$ are also equivalent.

Since in integral metrics the polynomials $B_nf$ cannot be used to approximate the function, Kantorovich suggested the following modification:

$$K_nf(x) = \sum_{k=0}^{n} \left((n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} f(u)\,du\right)\binom{n}{k} x^k(1-x)^{n-k}.$$

Apart from the saturation case $\alpha=1$, the integral analogue of the above results was not settled until very recently. In [12, 13] we gave the characterization of $\|K_nf-f\|_{L^p(0,1)} \leqslant Kn^{-\alpha}$ by means of first and second order

[1] $K$ always denotes a positive constant not necessarily the same at each occurrence.



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differences, and the aim of this article is to answer most of the conjectures and open problems which arose in this circle of problems.

The saturation case $\alpha = 1$ was settled by Maier [7, 8] and Riemenschneider [9], and further equivalent statements were found by Becker and Nessel [1] (in $L^1$), and Becker *et al.* [3] (in $L^p$, $p > 1$). In the following, $\|\cdot\|_{BV+C}$ denotes the sum of the total variation and the supremum norm, $F(x) = \int_0^x f(t)\,dt$,

$$\Delta_h^1(f; x) = f(x+h) - f(x),$$

and

$$\Delta_h^*(f; x) = xf(x-(1-x)h) - f(x) + (1-x)f(x+xh).$$

With these notations they proved

THEOREM A [7, 1]. *For $f \in L^1(0, 1)$ the following are equivalent*:

(i) $\|K_n f - f\|_{L^1(0,1)} \leqslant Kn^{-1}$; $(n = 1, 2, \ldots)$,

(ii) *$f$ is absolutely continuous*[2] *and $\varphi f' = \eta$ is of bounded variation on* $[0, 1]$ *with* $\eta(0) = \eta(1) = 0$;

(iii) $\|\varphi\Delta_h^2(F)\|_{BV+C[h,1-h]} \leqslant Kh^2$ $(h > 0)$;

(iv) $\|\Delta_h^*(F)\|_{BV+C[h/(1+h),1/(1+h)]} \leqslant Kh^2$ $(h > 0)$.

THEOREM B [3, 9]. *If $1 < p < \infty$ and $f \in L^p(0, 1)$, then the following statements are equivalent*:

(i) $\|K_n f - f\|_{L^p(0,1)} \leqslant Kn^{-1}$ $(n = 1, 2, \ldots)$;

(ii) *$f$ has an absolutely continuous derivative $f'$ with* $(\varphi f')' \in L^p(0, 1)$;

(iii) $\|(\varphi\Delta_h^2(F))'\|_{L^p(h,1-h)} \leqslant Kh^2$ $(h > 0)$;

(iv) $\|(\Delta_h^*(F))'\|_{L^p(h/(1+h),1/(1+h))} \leqslant Kh^2$ $(h > 0)$.

For $0 < \alpha < 1$ they stated

*Conjecture* 1 [1, 2]. If $0 < \alpha < 1$ and $p = 1$, then

$$\|K_n f - f\|_{L^p(0,1)} \leqslant Kn^{-\alpha} \qquad (n = 1, 2, \ldots) \tag{1.1}$$

is equivalent to

$$\|\varphi^\alpha \Delta_h^2(F)\|_{BV+C[h,1-h]} \leqslant Kh^{2\alpha} \qquad (h > 0). \tag{1.2}$$

[2] This naturally means that $f$ coincides a.e. with an absolutely continuous function.

*Conjecture* 2 [1]. For $0 < \alpha < 1$ and $p = 1$, (.1.1) is equivalent to

$$\|\varphi^{\alpha-1}\Delta_h^*(F)\|_{BV+C[h/(1+h),1/(1+h)]} \leqslant Kh^{2\alpha} \qquad (h > 0). \tag{1.3}$$

*Conjecture* 3 [3]. If $0 < \alpha < 1$ and $1 < p < \infty$, then (1.1) is equivalent to

$$\|(\varphi^{\alpha}\Delta_h^2(F))'\|_{L^p(h,1-h)} \leqslant Kh^{2\alpha} \qquad (h > 0). \tag{1.4}$$

*Conjecture* 4 [3]. For $0 < \alpha < 1$ and $1 < p < \infty$, (1.1) and

$$\|(\varphi^{\alpha-1}\Delta_h^*(F))'\|_{L^p(h/(1+h),1/(1+h))} \leqslant Kh^{2\alpha} \qquad (h > 0) \tag{1.5}$$

are equivalent.

In [12, 13] we proved

THEOREM C. *If* $1 \leqslant p < \infty$, $f \in L^p(0, 1)$ *and* $0 < \alpha < 1$, *then* (1.1) *is equivalent to*

$$\|\Delta^2_{h\sqrt{\varphi}}(f)\|_{L^p(h^2,1-h^2)} + h^{\alpha}\,\|\Delta_h^1(f)\|_{L^p(0,1-h)} \leqslant Kh^{2\alpha} \qquad (h > 0). \tag{1.6}$$

THEOREM D. *If* $1 < p < \infty$, $f \in L^p(0, 1)$ *and* $\alpha = 1$, *then* (1.1) *is equivalent to any of the following*:

(i) *$f$ has an absolutely continuous derivative with $\varphi f'' \in L^p(0, 1)$*;

(ii) $\|\varphi\Delta_h^2(f)\|_{L^p(h,1-h)} \leqslant Kh^2$ $(h > 0)$;

(iii) $\|\Delta^2_{h\sqrt{\varphi}}(f)\|_{L^p(h^2,1-h^2)} \leqslant Kh^2$ $(h > 0)$.

Concerning these results the following questions arise:

*Problem* 1 [12, 13]. Can we drop the second term in (1.6); i.e., for $1 \leqslant p < \infty$ and $0 < \alpha < 1$ is (1.1) equivalent to

$$\|\Delta^2_{h\sqrt{\varphi}}(f)\|_{L^p(h^2,1-h^2)} \leqslant Kh^{2\alpha} \qquad (h > 0)?$$

*Problem* 2 [12, 13]. For $1 \leqslant p < \infty$ and $0 < \alpha < 1$ is (1.1) equivalent to

$$\|\varphi^{\alpha}\Delta_h^2(f)\|_{L^p(h,1-h)} + h^{\alpha}\,\|\Delta_h^1(f)\|_{L^p(0,1-h)} \leqslant Kh^{2\alpha} \qquad (h > 0)? \tag{1.7}$$

On the positive real line the analogue of the Bernstein operator is the so-called Szász–Mirakjan operator, the integral-modification of which is

$$S_n^*f(x) = \sum_{k=0}^{\infty} \left(n\int_{k/n}^{(k+1)/n} f(u)\,du\right) e^{-nx}\,\frac{(nx)^k}{k!} \qquad (x \geqslant 0).$$

For these we proved in [11, 13]

THEOREM E. *If* $1 \leqslant p < \infty, f \in L^p(0, \infty)$ *and* $0 < \alpha < 1$, *then*

$$\|S_n^* f - f\|_{L^p(0,\infty)} \leqslant K n^{-\alpha} \qquad (n = 1, 2, \ldots) \tag{1.8}$$

*is equivalent to*

$$\|\Delta^2_{h\sqrt{\varphi_1}}(f)\|_{L^p(h^2,\infty)} + h^\alpha \|\Delta^1_h(f)\|_{L^p(0,\infty)} \leqslant K h^{2\alpha} \qquad (h > 0), \tag{1.9}$$

*where* $\varphi_1(x) = x$.

THEOREM F. *If* $1 < p < \infty$, $f \in L^p(0, \infty)$ *and* $\alpha = 1$, *then* (1.8) *is equivalent to any of the following conditions*:

(i) $f$ *has an absolutely continuous derivative with* $\varphi_1 f'' \in L^p(0, \infty)$;

(ii) $\|\cdot(f(\cdot) - 2f(\cdot + h) + f(\cdot + 2h))\|_{L^p(0,\infty)} \leqslant K h^2$ $(h > 0)$;

(iii) $\|\Delta^2_{h\sqrt{\varphi_1}}(f)\|_{L^p(h^2,\infty)} \leqslant K h^2$ $(h > 0)$.

Let us note that for $p = 1$ the analogue of Theorem A holds just as well for $S_n^*$ as can be seen from the considerations of [11].

For the operator $S_n^*$ we raised two problems:

*Problem* 3 [11, 13]. Can we replace (1.9) in Theorem E by

$$\|\Delta^2_{h\sqrt{\varphi_1}}(f)\|_{L^p(h^2,\infty)} \leqslant K h^{2\alpha} \qquad (h > 0)? \tag{1.10}$$

*Problem* 4 [11, 13]. Can we replace (1.9) in Theorem E by

$$\|\varphi_1^\alpha \Delta^2_h(f)\|_{L^p(h,\infty)} + h^\alpha \|\Delta^1_h(f)\|_{L^p(0,\infty)} \leqslant K h^{2\alpha} \qquad (h > 0)? \tag{1.11}$$

Now we answer the above conjectures and problems:

THEOREM 1. *If* $p = 1$, *then the answer to Problems* 1 *and* 3 *is positive.*

THEOREM 2. *Conjectures* 1–4 *are false and also the answer to Problems* 2 *and* 4 *is negative.*

Thus, the only undecided questions are Problems 1 and 3 in the case $1 < p < \infty$.[3]

*Remarks* 1. We shall prove that the answer to Conjectures 1 and 3 as well as to Problems 2 and 4 is negative for every $0 < \alpha < 1$ and $p \geqslant 1$. However, in the case of Conjectures 2 and 4 the proof is considerably simplified if we assume $\alpha < \frac{1}{2}$, so we shall disprove these conjectures only for $\alpha < \frac{1}{2}$. On the other hand, our opinion is that although the differences $\Delta_h^*$ are interesting, they are only of secondary importance, and Conjectures 2 and 4 are rather crude compared with Conjectures 1 or 3 (see the proofs below).

[3] *Note added in proof.* The answer to Problems 1 and 3 is positive for $1 < p < \infty$, as well.

2. We shall show that conditions (1.2), (1.4), (1.7), and (1.11) are not sufficient for (1.1) and (1.8), respectively. On the other hand, neither are they necessary as will be indicated at the end of the proof. Thus, in the nonoptimal case $0 < \alpha < 1$ the only suitable characterizing second difference seems to be $\Delta^2_{h\sqrt{\varphi(x)}}(f; x)$.

## 2. Proof of Theorem 1

We have to prove that if $\varphi(x) = x$ or $\varphi(x) = x(1-x)$, then

$$\|\Delta^2_{h\sqrt{\varphi}}(f)\|_{L^1(h^2, b(h))} \leqslant K h^{2\alpha} \tag{2.1}$$

implies

$$\|\Delta^1_h(f)\|_{L^1(0, b(\sqrt{h}))} \leqslant K h^{\alpha}, \tag{2.2}$$

where $b(h) = \infty$ if $\varphi(x) = x$, and $b(h) = 1 - h^2$ if $\varphi(x) = x(1-x)$.

First let us consider the case $\varphi(x) = x$ which corresponds to the Szász–Kantorovich operator. Let

$$f_h(x) = \frac{1}{2h} \int_{-h}^{h} f(x+t)\, dt.$$

For this we have

$$\begin{aligned} f_h(x) - f(x) &= \frac{1}{2h} \int_{-h}^{h} (f(x+t) - f(x))\, dt \\ &= \frac{\sqrt{x}}{2h} \int_0^{h/\sqrt{x}} (f(x + u\sqrt{x}) - 2f(x) + f(x - u\sqrt{x}))\, du, \end{aligned}$$

and if we assume (2.1), we have

$$\begin{aligned} \int_h^{\infty} |f_h(x) - f(x)|\, dx &\leqslant \sum_{i=0}^{\infty} \int_{2^i h}^{2^{i+1} h} \frac{\sqrt{x}}{2h} \int_0^{h/\sqrt{x}} |\Delta^2_{u\sqrt{x}}(f; x)|\, du\, dx \\ &\leqslant \sum_{i=0}^{\infty} \int_{2^i h}^{2^{i+1} h} \frac{\sqrt{2^{i+1}}}{\sqrt{h}} \int_0^{\sqrt{h/2^i}} |\Delta^2_{u\sqrt{x}}(f; x)|\, du\, dx \\ &\leqslant K \sum_{i=0}^{\infty} \frac{\sqrt{2^{i+1}}}{\sqrt{h}} \int_0^{\sqrt{h/2^i}} u^{2\alpha}\, du \leqslant K h^{\alpha} \sum_{i=0}^{\infty} 2^{-i\alpha} \leqslant K h^{\alpha}. \end{aligned} \tag{2.3}$$

This yields for $u \geqslant h$,

$$\left| \frac{1}{2h} \int_0^{2h} tf(u-h+t)\,dt - \int_h^{2h} f(u-h+t)\,dt \right|$$

$$= \left| \int_u^{\infty} (f_h(x) - f(x))\,dx \right| \leqslant Kh^{\alpha},$$

i.e.,

$$\left| \int_0^{\infty} f(u-h+t)\,X_h(t)\,dt \right| \leqslant Kh^{\alpha},$$

where

$$\begin{aligned} X_h(t) &= t/2h && \text{if} \quad 0 \leqslant t < h, \\ &= t/2h - 1 && \text{if} \quad h \leqslant t \leqslant 2h. \end{aligned} \tag{2.4}$$

Now

$$\begin{aligned} X_h(t) - \tfrac{1}{2}X_{h/2}(t) - \tfrac{1}{2}X_{h/2}(t-h) &= \tfrac{1}{2} && \text{if} \quad h/2 \leqslant t < h, \\ &= -1 && \text{if} \quad h \leqslant t < 3h/2, \end{aligned}$$

and we obtain from the previous estimate

$$\left| \int_{u-h/2}^{u} f(t)\,dt - \int_u^{u+h/2} f(t)\,dt \right|$$

$$\leqslant \left| \int_0^{\infty} f(u-h+t)\,2X_h(t)\,dt \right|$$

$$+ \left| \int_0^{\infty} f(u-h+t)\,X_{h/2}(t)\,dt \right|$$

$$+ \left| \int_0^{\infty} f(u-h+t)\,X_{h/2}(t-h)\,dt \right| \leqslant Kh^{\alpha};$$

i.e., with $2h$ instead of $h/2$,

$$|f_h(x) - f_h(x+2h)| \leqslant Kh^{\alpha-1} \qquad (x \geqslant 3h). \tag{2.5}$$

For arbitrary $a \geqslant 0$ we obtain from (2.3) and (2.5)

$$\int_{3h+a}^{6h+a} |f(x) - f(x+2h)|\,dx$$

$$\leqslant \int_{3h+a}^{6h+a} |f(x) - f_h(x)|\,dx + \int_{3h+a}^{6h+a} |f(x+2h) - f_h(x+2h)|\,dx$$

$$+ \int_{3h+a}^{6h+a} |f_h(x) - f_h(x+2h)|\,dx \leqslant Kh^{\alpha}$$

with a $K$ independent of $a$. This yields for $b \geqslant 0$

$$\int_{h+b}^{2h+b} |f(x)-f(x+h/2)|\,dx \leqslant Kh^\alpha, \quad \int_{h+b}^{2h+b} |f(x+h/2)-f(x+h)|\,dx \leqslant Kh^\alpha,$$

and adding these two inequalities we obtain

$$\int_{h+b}^{2h+b} |f(x)-f(x+h)|\,dx \leqslant Kh^\alpha \qquad (b \geqslant 0),$$

and together with this also

$$\int_{h}^{3h} |f(x)-f(x+h)|\,dx \leqslant Kh^\alpha. \tag{2.6}$$

Let

$$\omega(\delta) = \sup_{0<h\leqslant\delta} \|f(\cdot)-f(\cdot+h)\|_{L^1(h,\infty)}.$$

By (2.3) and (2.6)

$$\begin{aligned}
&\int_h^\infty |f(x)-f(x+h)|\,dx \\
&\quad\leqslant \int_h^{3h} |f(x)-f(x+h)|\,dx + \int_{3h}^\infty |f(x)-f_h(x)|\,dx \\
&\qquad+ \int_{3h}^\infty |f(x+h)-f_h(x+h)|\,dx + \int_{3h}^\infty |f_h(x)-f_h(x+h)|\,dx \\
&\quad\leqslant Kh^\alpha + \int_{3h}^\infty \int_0^h |f'_h(x+t)|\,dt\,dx \\
&\quad\leqslant Kh^\alpha + \int_0^h dt \int_{3h}^\infty \frac{1}{2h} |f(x+t+h)-f(x+t-h)|\,dx \\
&\quad\leqslant Kh^\alpha + h\frac{1}{2h}\,\omega(2h) \leqslant Kh^\alpha + \frac{1}{2}\,\omega(2h);
\end{aligned}$$

i.e., for $\omega$ we have

$$\omega(h) \leqslant Kh^\alpha + \tfrac{1}{2}\omega(2h).$$

Iterating this $k = [\log_2 1/h]$ times we get

$$\omega(h) \leqslant K\left(h^\alpha + \frac{1}{2}(2h)^\alpha + \cdots + \frac{1}{2^k}(2^k h)^\alpha\right) + \frac{1}{2^{k+1}}\,\omega(2^{k+1}h) \leqslant Kh^\alpha. \tag{2.7}$$

Now

$$\int_h^{2h} |f(x)|\,dx = \int_h^{\infty} |f(x)|\,dx - \int_h^{\infty} |f(x+h)|\,dx$$

$$\leqslant \int_h^{\infty} |f(x) - f(x+h)|\,dx \leqslant Kh^{\alpha},$$

and adding these for $h/2$, $h/4$, etc., we get $\int_0^h |f(x)|\,dx \leqslant Kh^{\alpha}$, and together with this also

$$\int_0^h |f(x) - f(x+h)|\,dx \leqslant Kh^{\alpha}. \tag{2.8}$$

We obtain (2.2) by adding (2.7) and (2.8), and the proof is complete in the case $\varphi(x) = x$.

Let us turn to Problem 1, to the weight $\varphi(x) = x(1-x)$. Exactly as above we get

$$|f_h(x) - f(x)| \leqslant \frac{\sqrt{\varphi(x)}}{2h} \int_0^{h/\sqrt{\varphi(x)}} |\Delta^2_{u\sqrt{\varphi(x)}}(f;x)|\,du,$$

and

$$\int_h^{1-h} |f_h(x) - f(x)|\,dx = \int_h^{1/2} + \int_{1/2}^{1-h} \leqslant Kh^{\alpha}.$$

Now there is a point $x_0 \in (\frac{1}{3}, \frac{2}{3})$ (e.g., a Lebesgue-point of $|f|$) for which

$$\frac{1}{h} \int_{x_0}^{x_0+h} |f(t)|\,dt \leqslant K \tag{2.9}$$

is satisfied. With the functions (2.4) we have (let $f(x) = 0$ for $x \notin [0, 1]$)

$$\int_u^{x_0} (f_h(x) - f(x))\,dx$$

$$= \int_0^{\infty} f(u-h+t)\,X_h(t)\,dt - \int_0^{\infty} f(x_0-h+t)\,X_h(t)\,dt$$

$$= \int_0^{\infty} f(u-h+t)\,X_h(t)\,dt + \mathcal{O}(h),$$

and since the left hand side is $\mathcal{O}(h^\alpha)$, we obtain for $h \leqslant \frac{1}{6}$ and $3h \leqslant u \leqslant 1 - 3h$

$$\left| \int_0^\infty f(u-h+t)\, X_h(t)\, dt \right| \leqslant K h^\alpha.$$

Using this and (2.9) the proof can be completed as above.

## 3. Proof of Theorem 2

For $0 < a < \frac{1}{2}$ and $\varepsilon < a^2/4$ let us consider the function (see Fig. 1)

$$\begin{aligned} f_{a,\varepsilon}(x) &= 0 && \text{if } \ |x-a| \geqslant \varepsilon \\ &= \varepsilon^{-2}(x-a+\varepsilon)^2 && \text{if } \ a-\varepsilon < x \leqslant a-\varepsilon/2 \\ &= \tfrac{1}{2} - \varepsilon^{-2}(x-a)^2 && \text{if } \ a-\varepsilon/2 < x \leqslant a+\varepsilon/2 \\ &= \varepsilon^{-2}(x-a-\varepsilon)^2 && \text{if } \ a+\varepsilon/2 < x \leqslant a+\varepsilon. \end{aligned} \tag{3.1}$$

Our counterexamples will be built up from the functions $f_{a,\varepsilon}$ with suitable $a$'s and $\varepsilon$'s.

Let $\varphi_1(x) = x$. We shall use the following estimates in which $K$ denotes absolute constants.

1. For $h^* = \varepsilon/\sqrt{a}$ we have

$$\Delta^2_{h^* \sqrt{\varphi_1(x)}}(f_{a,\varepsilon}; x) = 0 \qquad \text{if } \ x \notin (a-3\varepsilon, a+3\varepsilon), \tag{3.2}$$

and

$$\| \Delta^2_{h^* \sqrt{\varphi_1}}(f_{a,\varepsilon}) \| )\|_{L^p(h^2,\infty)} \geqslant \tfrac{1}{8}\varepsilon^{1/p}. \tag{3.3}$$

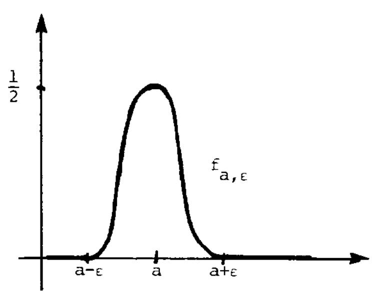


Fig. 1. The function $f_{a,\varepsilon}$.

In (3.3) use that $\Delta^2_{h^*\sqrt{\omega_1(x)}}(f_{a,\varepsilon};x) \geqslant \frac{1}{4}$ when $x \in ([\frac{1}{2}(\sqrt{h^{*2}+4a}+h^*)]^2, [\frac{1}{2}(\sqrt{h^{*2}+4(a+\varepsilon/2)}+h^*)]^2)$, i.e., $x - h^*\sqrt{x} \in (a, a+\varepsilon/2)$, and

$$[\tfrac{1}{2}(\sqrt{h^{*2}+4(a+\varepsilon/2)}+h^*)]^2 - [\tfrac{1}{2}(\sqrt{h^*+4a}+h^*)]^2 \geqslant \varepsilon/2.$$

2. Furthermore,

$$\|\varphi_1^\alpha \Delta_h^2(f_{a,\varepsilon})\|_{L^p(h,\infty)} \leqslant K \begin{cases} h^\alpha \varepsilon^{1/p} & \text{if } a \leqslant h \\ a^\alpha \varepsilon^{1/p} & \text{if } \varepsilon \leqslant h \leqslant a \\ a^\alpha \varepsilon^{1/p-2} h^2 & \text{if } 0 < h \leqslant \varepsilon. \end{cases}$$

Indeed, the second estimate follows from the fact that $\Delta_h^2(f_{a,\varepsilon};x) = 0$ if $|x-a| > h+\varepsilon$, and in the last line use $|f''_{a,\varepsilon}(x)| \leqslant 2\varepsilon^{-2}$ (a.e.).

3. Moreover,

$$\|\Delta_h^1(f_{a,\varepsilon})\|_{L^p(0,\infty)} \leqslant K \begin{cases} \varepsilon^{1/p} & \text{if } \varepsilon \leqslant h \\ \varepsilon^{1/p-1} h & \text{if } 0 < h \leqslant \varepsilon. \end{cases} \tag{3.4}$$

Here one has to use $|f'_{a,\varepsilon}(x)| \leqslant \varepsilon^{-1}$.

4. Let $F_{a,\varepsilon}(x) = \int_0^x f_{a,\varepsilon}(t)\,dt$. Clearly, $|F_{a,\varepsilon}| \leqslant \varepsilon$, hence we have the estimates

$$|\varphi_1^\alpha(x)\,\Delta_h^2(F_{a,\varepsilon};x)| \leqslant K \begin{cases} 0 & \text{if } a+\varepsilon+h \leqslant x \\ \varepsilon\varphi_1^\alpha(x) & \text{if } \varepsilon \leqslant h \\ a^\alpha \varepsilon^{-1} h^2 & \text{if } h \leqslant \varepsilon. \end{cases} \tag{3.5}$$

I. We have a counterexample for Problem 4. Let $n > 0$ be an integer, $0 < a < \frac{1}{2}$, $\eta$ a small positive number, and $a_i = a_i^{(n)} = a/2^i$ $(i = 1, 2, \ldots, n)$. Suppose $\eta$ is so small that $\eta < a_n/2$ is satisfied.

Let $\varepsilon_i = \eta\sqrt{a_i}$ $(i = 1, 2, \ldots, n)$ and

$$g(x) = g_{n,a,\eta}(x) = \sum_{i=1}^n \eta^{2\alpha} \varepsilon_i^{-1/p} f_{a_i,\varepsilon_i}(x).$$

Clearly,

$$\text{support } g \subseteq \left(\frac{a}{2^n} - \eta\sqrt{\frac{a}{2^n}}, a\right),$$

and

$$\|g_{n,a,\eta}\|_{L^p(0,\infty)} \leqslant \sum_{i=1}^n \eta^{2\alpha} \varepsilon_i^{-1/p} \|f_{a_i,\varepsilon_i}\|_{L^p(0,\infty)}$$

$$\leqslant \sum_{i=1}^n \eta^{2\alpha} \varepsilon_i^{-1/p} \varepsilon_i^{1/p} \leqslant n\eta^{2\alpha} \leqslant n2^{-2\alpha n} \leqslant K. \tag{3.6}$$

If $\eta$ is sufficiently small, then $a_{i+1} + 3\varepsilon_{i+1} < a_i - 3\varepsilon_i$ $(i = 1,..., n-1)$, and in this case we get from point 1 above

$$\|\Delta^2_{\eta\sqrt{\varphi_1}}(g_{n,a,\eta})\|_{L^p(\eta^2,\infty)} \geqslant \tfrac{1}{8} n^{1/p}\eta^{2\alpha}. \tag{3.7}$$

Point 2 gives

$$\|\varphi_1^\alpha \Delta_h^2(\eta^{2\alpha}\varepsilon_i^{-1/p} f_{a_i,\varepsilon_i})\|_{L^p(h,\infty)} \leqslant K \begin{cases} \eta^{2\alpha}h^\alpha \leqslant \eta^\alpha h^{2\alpha} & \text{if } \ a_i < h \\ a_i^\alpha \eta^{2\alpha} & \text{if } \ \varepsilon_i \leqslant h \leqslant a_i \\ \varepsilon_i^{2\alpha-2}h^2 & \text{if } \ 0 < h \leqslant \varepsilon_i. \end{cases}$$

Using this we show that for $h > 0$

$$\|\varphi_1^\alpha \Delta_h^2(g_{n,a,\eta})\|_{L^p(h,\infty)} \leqslant \sum_{i=1}^{n} \|\varphi_1^\alpha \Delta_h^2(\eta^{2\alpha}\varepsilon_i^{-1/p} f_{a_i,\varepsilon_i})\|_{L^p(h,\infty)} \leqslant Kh^{2\alpha}. \tag{3.8}$$

In fact, if $h \geqslant \varepsilon_1 = \eta\sqrt{a_1}$, then the sum in the middle of (3.8) is at most

$$K\eta^{2\alpha} \sum_{i=1}^{n} a_i^\alpha + Kn\eta^\alpha h^{2\alpha} \leqslant K\eta^{2\alpha}a^\alpha + Kh^{2\alpha} \leqslant Kh^{2\alpha}.$$

If $\varepsilon_{i+1} \leqslant h < \varepsilon_i$ $(i = 1,..., n-1)$, then the sum in question is at most

$$K \sum_{j=i+1}^{n} a_j^\alpha \eta^{2\alpha} + K \sum_{j=1}^{i} \varepsilon_j^{2\alpha-2}h^2 \leqslant Ka_{i+1}^\alpha \eta^{2\alpha} + K\varepsilon_i^{2\alpha-2}h^2 \leqslant Kh^{2\alpha},$$

and we can argue similarly when $h \leqslant \varepsilon_n$ to deduce

$$\|\varphi_1^\alpha \Delta_h^2(g_{n,a,\eta})\|_{L^p(h,\infty)} \leqslant K \sum_{j=1}^{n} \varepsilon_j^{2\alpha-2}h^2 \leqslant K\varepsilon_n^{2\alpha-2}h^2 \leqslant Kh^{2\alpha}.$$

Our next aim is to estimate $\|\Delta_h^1(g_{n,a,\eta})\|_{L^p(0,\infty)}$. For $\varepsilon_n \leqslant h$ we obtain from point 3

$$\begin{aligned} \|\Delta_h^1(g_{n,a,\eta})\|_{L^p(0,\infty)} &\leqslant \sum_{i=1}^{n} \eta^{2\alpha}\varepsilon_i^{-1/p} \|\Delta_h^1(f_{a_i,\varepsilon_i})\|_{L^p(h,\infty)} \\ &\leqslant Kn\eta^{2\alpha} \leqslant Kn\eta^\alpha(\varepsilon_n/\sqrt{a_n})^\alpha \\ &\leqslant K(n\eta^\alpha a_n^{-\alpha/2})\, h^\alpha \leqslant Kh^\alpha. \end{aligned}$$

For $h < \varepsilon_n$ we have

$$\begin{aligned} \|\Delta_h^1(g_{n,a,\eta})\|_{L^p(0,\infty)} &\leqslant K \sum_{i=1}^{n} \eta^{2\alpha}\varepsilon_i^{-1}h \leqslant K\eta^{2\alpha}\varepsilon_n^{-1}h \\ &\leqslant K\eta^\alpha a_n^{-\alpha/2}\varepsilon_n^{\alpha-1}h \leqslant Kh^\alpha, \end{aligned}$$

and this together with the previous estimate gives for all $h>0$

$$\|\Delta_h^1(g_{n,a,\eta})\|_{L^p(0,\infty)} \leqslant Kh^\alpha. \tag{3.9}$$

Let us now turn to the construction of the counterexample. Let $(n_i, a_i, \eta_i)$ $(i=1,2,\ldots)$ be a sequence of triplets for which the above estimates hold (i.e., $\{\eta_i\}$ decreases sufficiently rapidly), and for which

$$\sum_{j=i+1}^{\infty} n_j^{-1/2p} \leqslant \eta_i^{2\alpha} \qquad (i=1,2,\ldots).$$

Since $g'_{n_i,a_i,\eta_i}$ is absolutely continuous, $|g''_{n_i,a_i,\eta_i}| \leqslant K_i$, and $g_{n_i,a_i,\eta_i}(x)=0$ for $x \leqslant a_i/2^n - \eta\sqrt{a_i/2^n}$, there are constants $A_i$ with

$$\|\Delta^2_{h\sqrt{\varphi_1}}(g_{n_i,a_i,\eta_i})\|_{L^p(h^2,\infty)} \leqslant A_i h^2.$$

We may also suppose that

$$\sum_{j=1}^{i-1} A_j n_j^{-1/2p} \leqslant n_i^{1/2p} \qquad (i=2,3,\ldots).$$

Let now

$$f(x)=\sum_{i=1}^{\infty} n_i^{-1/2p} g_{n_i,a_i,\eta_i}(x),$$

where we may suppose (by appropriate choice of $\{(n_i,a_i,\eta_i)\}_{i=1}^{\infty}$) that if $(p'_i, q'_i)$ is the smallest interval which contains the support of $g_{n_i,a_i,\eta_i}$ and $p_i = p'_i - 3\eta_i$, $q_i = q'_i + 3\eta_i$, then the inequalities $q_i + \eta_i\sqrt{q_i} < p'_{i-1}$ and $q'_{i+1} < p_i - \eta_i\sqrt{q_i}$ are satisfied for all $i \geqslant 2$. This gives for $x \in (p_i, q_i)$

$$\Delta^2_{\eta_i\sqrt{\varphi_1(x)}}(f;x) = \Delta^2_{\eta_i\sqrt{\varphi_1(x)}}(n_i^{-1/2p} g_{n_i,a_i,\eta_i};x). \tag{3.10}$$

By (3.6) we have

$$\|f\|_{L^p(0,\infty)} \leqslant K\sum_{i=1}^{\infty} n_i^{-1/p} < \infty,$$

i.e., $f \in L^p(0,\infty)$. Now (3.8) and (3.9) give

$$\|\varphi_1^\alpha \Delta_h^2(f)\|_{L^p(h,\infty)} \leqslant K\sum_{i=1}^{\infty} n_i^{-1/2p} h^{2\alpha} \leqslant Kh^{2\alpha},$$

$$\|\Delta_h^1(f)\|_{L^p(0,\infty)} \leqslant K\sum_{i=1}^{\infty} n_i^{-1/2p} h^{\alpha} \leqslant Kh^{\alpha}.$$

However, taking into account (3.2), (3.10), and (3.7), we obtain for $h=\eta_i$ $(i\geqslant 2)$

$$\begin{aligned}
&\|\Delta^2_{h\sqrt{\varphi_1}}(f)\|_{L^p(h^2,\infty)}\\
&\quad\geqslant \|\Delta^2_{h\sqrt{\varphi_1}}(n_i^{-1/2p}g_{n_i,a_i,\eta_i})\|_{L^p(h^2,\infty)}\\
&\qquad-\left(\sum_{j=1}^{i-1}+\sum_{j=i+1}^{\infty}\right)\|\Delta^2_{h\sqrt{\varphi_1}}(n_j^{-1/2p}g_{n_j,a_j,\eta_j})\|_{L^p(h^2,\infty)}\\
&\quad\geqslant\frac{1}{8}n_i^{1/2p}h^{2\alpha}-\sum_{j=1}^{i-1}A_jn_j^{-1/2p}h^2-K\sum_{j=i+1}^{\infty}\|n_j^{-1/2p}g_{n_j,a_j,\eta_j}\|_{L^p(0,\infty)}\\
&\quad\geqslant\frac{1}{8}n_i^{1/2p}h^{2\alpha}-n_i^{1/2p}h^2-K\sum_{j=i+1}^{\infty}n_j^{-1/2p}\geqslant\frac{1}{8}n_i^{1/2p}h^{2\alpha}\\
&\qquad-n_i^{1/2p}h^2-K\eta_i^{2\alpha}\geqslant\frac{1}{10}n_i^{1/2p}h^{2\alpha}
\end{aligned}$$

for sufficiently large $i$, i.e.,

$$\|\Delta^2_{h\sqrt{\varphi_1}}(f)\|_{L^p(h^2,\infty)}\neq\mathcal{O}(h^{2\alpha}).$$

Thus, for our $f$, (1.11) is satisfied but (1.9) does not hold. This proves, by Theorem E, that condition (1.11) is not sufficient for (1.8) to hold.

II. For Problem 2 one can argue as above with $\varphi(x)=x(1-x)$ instead of $\varphi_1(x)=x$. Clearly, $x(1-x)$ and $x$ behave similarly in the neighborhoods of the origin (the factor $(1-x)$ does not play any role), thus the considerations of point I above work also for $\varphi$. The necessary changes are obvious.

III. Since we proved above that condition (1.7) is not sufficient for (1.1), to disprove Conjecture 3 it is enough to show that (1.7) implies (1.4). In fact,

$$(\varphi^\alpha(x)\,\Delta_h^2(F;x))'=\varphi^\alpha(x)\,\Delta_h^2(f;x)+\alpha\varphi^{\alpha-1}(x)(1-2x)\times\int_0^h(f(x+t)-f(x-h+t))\,dt,$$

and so (1.7) gives

$$\begin{aligned}
&\|(\varphi^\alpha\Delta_h^2(F))'\|_{L^p(h,1-h)}\\
&\quad\leqslant\|\varphi^\alpha\Delta_h^2(f)\|_{L^p(h,1-h)}+h^{\alpha-1}\left\|\int_0^h|f(\cdot+t)-f(\cdot-h+t)|\,dt\right\|_{L^p(h,1-h)}\\
&\quad\leqslant Kh^{2\alpha}+Kh^{\alpha-1}\int_0^h\|f(\cdot+t)-f(\cdot-h+t)\|_{L^p(h,1-h)}\,dt\\
&\quad\leqslant Kh^{2\alpha}+Kh^{\alpha-1}\int_0^h h^\alpha\,dt\leqslant Kh^{2\alpha},
\end{aligned}$$

where we used that the $L^p$-norm of an integral is not greater than the integral of the corresponding norms (see [10, p. 271]).

IV. Let us consider Conjecture 1. Here $p=1$, and first we show that for the function $f$ given in II (see also I) we have

$$|\varphi^\alpha(x)\Delta_h^2(F;x)| \leqslant Kh^{2\alpha} \qquad (h \leqslant x \leqslant 1-h), \tag{3.11}$$

where $F(x)=\int_0^x f(t)\,dt$. Clearly, it is enough to prove the analoguous relation for the function $f$ constructed effectively from I.

By (3.5) we have (see I)

$$\|\varphi_1^\alpha \Delta_h^2(F_{a/2^i,\varepsilon_i}\eta^{2\alpha}\varepsilon_i^{-1})\|_{C(h,\infty)} \leqslant K\eta^{2\alpha}(a/2^i)^\alpha + K\eta^{2\alpha}h^\alpha;$$

hence we obtain for $\varepsilon_{i+1} < h \leqslant \varepsilon_i$ and $G_{n,a,\eta}=\int_0^x g_{n,a,\eta}(t)\,dt$ (see also I)

$$\begin{aligned}
&\|\varphi_1^\alpha \Delta_h^2(G_{n,a,\eta})\|_{C(h,\infty)}\\
&\quad\leqslant K\sum_{j=i+1}^{n}\eta^{2\alpha}(a/2^j)^\alpha + K\sum_{j=i+1}^{n}\eta^{2\alpha}h^\alpha + K\sum_{j=1}^{i}\varepsilon_j^{2\alpha-2}h^2\\
&\quad\leqslant K\eta^{2\alpha}(a/2^{i+1})^\alpha + K\eta^\alpha(a/2^{i+1})^{-\alpha/2}\,n\varepsilon_{i+1}^\alpha h^\alpha + K\varepsilon_i^{2\alpha-2}h^2 \leqslant Kh^{2\alpha}
\end{aligned}$$

(take into account that $\varepsilon_i=\eta\sqrt{a/2^i}$). A similar estimate can be given for $h \leqslant \varepsilon_n$. Furthermore for $h \geqslant \varepsilon_1$, we have

$$\begin{aligned}
\|\varphi_1^\alpha \Delta_h^2(G_{n,a,\eta})\|_{C(h,\infty)} &\leqslant K\eta^{2\alpha}\sum_{i=1}^{n}(a/2^i)^\alpha + Kn\eta^{2\alpha}h^\alpha\\
&\leqslant K\eta^{2\alpha}a^\alpha + Kn\eta^\alpha(a/2)^{-\alpha/2}\,\varepsilon_1^\alpha h^\alpha \leqslant Kh^{2\alpha}.
\end{aligned}$$

The above inequalities give

$$\|\varphi_1^\alpha \Delta_h^2(F)\|_{C(h,\infty)} \leqslant K\sum_{i=1}^{\infty} n_i^{-1/2p}h^{2\alpha} \leqslant Kh^{2\alpha}$$

as was stated above.

Now $F(x)=\int_0^x f(t)\,dt$ is absolutely continuous; hence exactly as in III we obtain from (1.7)

$$\|\varphi^\alpha \Delta_h^2(F)\|_{BV(h,1-h)} = \|(\varphi^\alpha \Delta_h^2(F))'\|_{L^1(h,1-h)} \leqslant Kh^{2\alpha}.$$

This and (3.11) show that for this $f$, (1.2) holds but (1.6) is not satisfied; hence, by Theorem C, (1.2) is not a sufficient condition for (1.1).

V. We disprove Conjecture 4 for $\alpha < \frac{1}{2}$ and $p > 1$. For the second difference $\Delta_h^*$ and for $\varphi(x) = x(1-x)$ we have

$$\begin{aligned}
&(\varphi^{\alpha-1}(x)\,\Delta_h^*(F;x))' \\
&\quad = (\alpha-1)\,\varphi^{\alpha-2}(x)(1-2x)\,\Delta_h^*(F;x) \\
&\qquad + \varphi^{\alpha-1}(x)(F(x-(1-x)h) - F(x+hx)) \\
&\qquad + \varphi^{\alpha-1}(x)(xf(x-h(1-x)) - f(x) + (1-x)f(x+hx)) \\
&\qquad + \varphi^{\alpha-1}(x)\,h(xf(x-h(1-x)) + (1-x)f(x+hx)).
\end{aligned}$$

Let us consider the functions $f_{a,\varepsilon}$ from (3.1) ($\varepsilon < a^2/4$), and let $F_{a,\varepsilon}(x) = \int_0^x f_{a,\varepsilon}(t)\,dt$, $h_1 = 2\varepsilon/a$. A simple calculation gives

$$\begin{aligned}
&\|\varphi^{\alpha-2}\Delta_{h_1}^*(F_{a,\varepsilon})\|_{L^p(h_1/(1+h_1),\,1/(1+h_1))} \\
&\qquad \leqslant K a^{\alpha-2}\varepsilon h_1^{1/p} \leqslant K a^{\alpha-2-1/p}\varepsilon^{1+1/p}, \\
&\|\varphi^{\alpha-1}(F(\cdot-(1-\cdot\,)\,h_1) - F(\cdot+h_1\cdot))\|_{L^p(h_1/(1+h_1),\,1/(1+h_1))} \\
&\qquad \leqslant K a^{\alpha-1}\varepsilon h_1^{1/p} \leqslant K a^{\alpha-1-1/p}\varepsilon^{1+1/p}, \\
&\|\varphi^{\alpha-1}h_1(\cdot f(\cdot-h_1(1-\cdot\,)) + (1-\cdot\,)f(\cdot+h_1\cdot))\|_{L^p(h_1/(1+h_1),\,1/(1+h_1))} \\
&\qquad \leqslant K a^{\alpha-1}h_1\varepsilon^{1/p} \leqslant K a^{\alpha-2}\varepsilon^{1+1/p}, \\
&\|\varphi^{\alpha-1}(\cdot f(\cdot-h_1(1-\cdot\,)) - f + (1-\cdot\,)f(\cdot+h_1\cdot))\|_{L^p(h_1/(1+h_1),\,1/(1+h_1))} \\
&\qquad \geqslant \left\{\int_{a-2\varepsilon}^{a-(3/2)\varepsilon} \left|\varphi^{\alpha-1}(x)(1-x)f\left(x+\frac{2\varepsilon}{a}x\right)\right|^p dx\right\}^{1/p} \geqslant c a^{\alpha-1}\varepsilon^{1/p}
\end{aligned}$$

with a $c > 0$ independent of $a$ and $\varepsilon$.

These imply

$$\|(\varphi^{\alpha-1}\Delta_{h_1}^*(F_{a,\varepsilon}))'\|_{L^p(h_1/(1+h_1),\,1/(1+h_1))} \geqslant c_1 a^{\alpha-1}\varepsilon^{1/p} \geqslant \frac{c_1}{2} a^{3\alpha-1}\varepsilon^{1/p-2\alpha}h_1^{2\alpha} \tag{3.12}$$

provided $\varepsilon$ is sufficiently small compared to $a$. At the same time we obtain from (3.4) and (3.13) (see below)

$$\|\Delta_{h\sqrt{\varphi}}^2(f_{a,\varepsilon})\|_{L^p(h^2,1-h^2)} + h^\alpha\,\|\Delta_h^1(f_{a,\varepsilon})\|_{L^p(0,1-h)} \leqslant K a^\alpha \varepsilon^{1/p-2\alpha}h^{2\alpha}$$

for every $h > 0$.

Thus, putting

$$g_{a,\varepsilon}(x) = a^{-\alpha}\varepsilon^{2\alpha-1/p}f_{a,\varepsilon}(x), \qquad G_{a,\varepsilon}(x) = \int_0^x g_{a,\varepsilon}(t)\,dt, \qquad h_1 = 2\varepsilon/a,$$

we have

$$\|g_{a,\varepsilon}\|_{L^p(0,1)} = a^{-\alpha}\varepsilon^{2\alpha-1/p}\|f_{a,\varepsilon}\|_{L^p(0,1)} \leqslant a^{-\alpha}\varepsilon^{2\alpha} \leqslant a^{3\alpha},$$

$$\|\Delta^2_{h\sqrt{\varphi}}(g_{a,\varepsilon})\|_{L^p(h^2,1-h^2)} + h^\alpha \|\Delta^1_h(g_{a,\varepsilon})\|_{L^p(0,1-h)} \leqslant Kh^{2\alpha} \qquad (h>0),$$

$$\|(\varphi^{\alpha-1}\Delta^*_{h_1}(G_{a,\varepsilon}))'\|_{L^p(h_1/(1+h_1),1/(1+h_1))} \geqslant \frac{c_1}{2}a^{2\alpha-1}h_1^{2\alpha}.$$

Let

$$f(x) = \sum_{i=1}^{\infty} a_i^{(1-2\alpha)/2} g_{a_i,\varepsilon_i}(x).$$

If the sequences $\{a_i\}$, $\{\varepsilon_i\}$ decrease sufficiently rapidly, then we get from $2\alpha < 1$ and from the previous estimates that $f \in L^p(0,1)$, $f$ satisfies condition (1.6) but it does not satisfy (1.5) (see also the argument of I above); hence, by Theorem C, condition (1.5) is not necessary for (1.1).

VI. Concerning Conjecture 2, we shall show that for a function $f$ satisfying (1.6)

$$\|\varphi^{\alpha-1}\Delta^*_h(F)\|_{C(h/(1+h),1/(1+h))} \neq \mathcal{O}(h^{2\alpha}),$$

so that condition (1.3) is not necessary for (1.1). We follow the argument of the previous point. With $h_1 = 2\varepsilon/a$ we have $a-\varepsilon > a/(1+h_1)$, so

$$\begin{aligned}\varphi^{\alpha-1}&\left(\frac{a}{1+h_1}\right)\Delta^*_{h_1}\left(F_{a,\varepsilon};\frac{a}{1+h_1}\right)\\ &=\varphi^{\alpha-1}\left(\frac{a}{1+h_1}\right)\left(1-\frac{a}{1+h_1}\right)F_{a,\varepsilon}\left(\frac{a}{1+h_1}+h_1\frac{a}{1+h_1}\right)\\ &\geqslant ca^{\alpha-1}\varepsilon \qquad (c>0)\end{aligned}$$

i.e.,

$$\|\varphi^{\alpha-1}\Delta^*_{h_1}(F_{a,\varepsilon})\|_{C(h_1/(1+h_1),1/(1+h_1))} \geqslant \frac{C}{2}a^{3\alpha-1}\varepsilon^{1-2\alpha}h_1^{2\alpha},$$

and we can argue as above in V (see the analogous inequality (3.12)).

VII. Finally, let us show that condition (1.11) is not only not sufficient but it is neither necessary for (1.8). The same argument shows that (1.7) is not necessary for (1.1), and since $\|\Delta^1_h(f)\|_{L^p(0,1-h)} \leqslant Kh^\alpha$ is also satisfied below, we obtain at the same time that neither (1.2) nor (1.4) is necessary for (1.1) (see the argument of III above).

The idea of point I was to keep the quotient $\varepsilon_i/\sqrt{a_i}$ fixed, where roughly speaking,

$$\|\Delta^2_{h\sqrt{\varphi_1}}(f_{a_i,\varepsilon_i})\|_{L^p(h^2,\infty)}/h^{2\alpha}$$

has its maximum. Here we fix $\varepsilon$ around which the quotient

$$\|\varphi_1^\alpha \Delta_h^2(f_{a,\varepsilon})\|_{L^p(h,\infty)}/h^{2\alpha}$$

attains its maximum.

Thus, for $n > 0$, $a > 0$, $\varepsilon > 0$, let

$$f_{n,a,\varepsilon}(x) = \sum_{i=1}^{n} a_i^{-\alpha} \varepsilon^{2\alpha - 1/p} f_{a_i,\varepsilon}(x), \qquad a_i = a/2^i \qquad (i = 1, 2, \ldots, n)$$

(the definition of $f_{a,\varepsilon}$ was given in (3.1)). For the functions $f_{a,\varepsilon}$ a simple calculation gives

$$\|\Delta^2_{h\sqrt{\varphi_1}}(f_{a,\varepsilon})\|_{L^p(h^2,\infty)} \leqslant K \begin{cases} \varepsilon^{1/p} & \text{if} \quad \varepsilon/\sqrt{a} \leqslant h \\ h^2 \varepsilon^{1/p-2} a & \text{if} \quad h < \varepsilon/\sqrt{a}, \end{cases} \tag{3.13}$$

$$\Delta_\varepsilon^2(f_{a,\varepsilon}; x) = 0 \qquad \text{if} \quad |x - a| \geqslant 2\varepsilon, \tag{3.14}$$

$$\|\varphi_1^\alpha \Delta_\varepsilon^2(f_{a,\varepsilon})\|_{L^p(\varepsilon,\infty)} \geqslant c a^\alpha \varepsilon^{1/p} \qquad (c > 0). \tag{3.15}$$

Using these we show that

$$\|\Delta^2_{h\sqrt{\varphi_1}}(f_{n,a,\varepsilon})\|_{L^p(h^2,\infty)} \leqslant K h^{2\alpha}, \qquad \|\Delta_h^1(f_{n,a,\varepsilon})\|_{L^p(0,\infty)} \leqslant K h^\alpha.$$

In fact, for $\varepsilon/\sqrt{a_{i+1}} \leqslant h < \varepsilon/\sqrt{a_i}$ $(i = 1, 2, \ldots, n-1)$

$$\begin{aligned} \|\Delta^2_{h\sqrt{\varphi_1}}(f_{n,a,\varepsilon})\|_{L^p(h^2,\infty)} &\leqslant \sum_{j=1}^{n} a_j^{-\alpha} \varepsilon^{2\alpha - 1/p} \|f_{a_j,\varepsilon}\|_{L^p(h^2,\infty)} \\ &\leqslant K \sum_{j=i+1}^{n} \varepsilon^{2\alpha} a_j^{-\alpha} + K \sum_{j=1}^{i} \varepsilon^{2\alpha-2} a_i^{1-\alpha} h^2 \\ &\leqslant K \varepsilon^{2\alpha} a_{i+1}^{-\alpha} + K h^2 \varepsilon^{2\alpha-2} a_i^{1-\alpha} \leqslant K h^{2\alpha}, \end{aligned}$$

and similar estimates hold for $h \geqslant \varepsilon/\sqrt{a_1}$ or $h \leqslant \varepsilon/\sqrt{a_n}$; furthermore, by (3.4)

$$\|\Delta_h^1(f_{n,a,\varepsilon})\|_{L^p(0,\infty)} \leqslant K \begin{cases} \displaystyle\sum_{i=1}^{n} a_i^{-\alpha} \varepsilon^{2\alpha} \leqslant K 2^{n\alpha} a^{-\alpha} \varepsilon^\alpha h^\alpha \leqslant K h^\alpha & \text{if} \quad \varepsilon \leqslant h \\ \displaystyle\sum_{i=1}^{n} a_i^{-\alpha} \varepsilon^{2\alpha-1} h \leqslant K 2^{n\alpha} a^{-\alpha} \varepsilon^\alpha h^\alpha \leqslant K h^\alpha & \text{if} \quad h \leqslant \varepsilon, \end{cases}$$

provided $\varepsilon$ is so small that $\varepsilon^a 2^{na}/a^\alpha < 1$. On the other hand, (3.14) and (3.15) give (we may assume $a_i - a_{i+1} > 4\varepsilon$ for $i = 1, 2,..., n-1$)

$$\|\varphi_1^\alpha \Delta_\varepsilon^2(f_{n,a,\varepsilon})\|_{L^p(\varepsilon,\infty)} \geqslant cn^{1/p}\varepsilon^{2\alpha}.$$

Using hese estimates the same method which was applied in I shows that for an appropriate choice of $n_i$, $a_i$, $\varepsilon_i$ the function

$$f(x) = \sum_{n=1}^{\infty} n_i^{-1/2p} f_{n_i, a_i, \varepsilon_i}(x)$$

will satisfy (1.9) but not (1.11).

The proof is complete.

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